Lorentz Transformations beyond Injectivity: The Ziegelstein Gedankenexperiment and the Emergence of Multi-Sheet Spacetime

From the Bricks Paradox to Multi-Sheet Spacetime Structure

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Abstract

Standard Lorentz transformations assume implicitly that the map $\tau \mapsto X^0(\tau)$ is injective: each proper-time instant corresponds to a unique coordinate time. We show that this assumption fails for worldlines with Lorentz factor $\gamma > \gamma_{\text{crit}}$, where the worldline intersects any fixed-t hypersurface Σ_t in $N > 1$ distinct spatial points. In this regime the standard boost $x' = \gamma(x - vt)$ must be replaced by a set of N extended transformations

$$x'_n = \gamma(x - vt) + \Phi_n, \quad n = 1, \ldots, N,$$

where Φ_n are topological phase offsets determined by the winding structure of the worldline through Σ_t. We derive these extended transformations from first principles, show that they preserve all conservation laws via the Ontological Identity Principle, and prove that they reduce to the standard Lorentz boost in the injective limit $N \to 1$. The physical origin of the extended structure is illustrated by the *Bricks Paradox*: a concrete relativistic thought experiment in which a single set of physical objects appears simultaneously at two causally separated locations, without any violation of special relativity. The extended transformations are shown to be the kinematic foundation of the De Giuseppe Qubit (DGQ) multi-sheet Hilbert space and the non-injectivity condition proved necessary and sufficient for finite holographic spacetime in the companion paper [1].

1 Introduction

Special relativity rests on the Lorentz transformation as the fundamental kinematic map between inertial frames. In its standard form, the boost along the x-axis reads:

$$t' = \gamma\left(t - \frac{v}{c^2}x\right), \tag{1}$$

$$x' = \gamma(x - vt), \tag{2}$$

$$y' = y, \quad z' = z, \tag{3}$$

with $\gamma = (1 - v^2/c^2)^{-1/2}$. This map is a bijection: it assigns to each spacetime event (t, x) a unique image (t', x').

The bijectivity of the Lorentz transformation rests on an implicit assumption that is almost never stated explicitly: the worldline $X^\mu(\tau)$ of any physical system maps *injectively* onto the coordinate time t of any inertial observer,

$$\tau_1 \neq \tau_2 \implies X^0(\tau_1) \neq X^0(\tau_2). \tag{4}$$

In other words, each moment of proper time corresponds to a distinct moment of coordinate time.

We show that this assumption fails when the Lorentz factor γ exceeds a critical threshold γ_{crit}. Above this threshold, the worldline is geometrically compressed relative to the simultaneity foliation $\{\Sigma_t\}$ to such a degree that a single worldline intersects each hypersurface Σ_t in $N > 1$ distinct spatial points. The standard Lorentz transformation, which assigns a unique image to each event, becomes inadequate: it must be replaced by a set of N extended transformations, one per intersection.

The physical content of this mathematical extension is captured by the *Bricks Paradox*, a thought experiment introduced in Section 2. The paradox reveals that non-injectivity is not a mathematical curiosity but a physical phenomenon with observable consequences — consequences that are already encoded, but invisible, in the standard formalism.

The extended Lorentz transformations derived here are the kinematic foundation of the De Giuseppe Qubit (DGQ) framework [2] and the worldline non-injectivity theorem of [1]. The present paper makes this kinematic foundation explicit and self-contained.

2 The Bricks Paradox

2.1 Narrative formulation

Alice and Bob are friends having dinner at point O at time $t = 0$. Bob works as a bricklayer for a construction company located at point Y, at coordinate distance L from O. During dinner, Bob and his company make a plan: they will build a wall at point $M = L/2$, the exact midpoint between O and Y, exactly 5 years from now (in Bob's rest frame).

At the end of dinner, Alice departs from O aboard a spacecraft carrying bricks destined for the construction company at Y. The spacecraft travels at ultra-relativistic velocity $v \approx c$, strictly subluminal.

The two characteristic time scales of the journey are:

$$\Delta t_{\text{obs}} = 10\,\text{years} \quad \text{(coordinate time elapsed for Bob, stationary)}, \tag{5}$$

$$\Delta\tau_{\text{prop}} = 4\,\text{months} \quad \text{(proper time elapsed for Alice on the spacecraft)}. \tag{6}$$

The Lorentz factor relating the two frames is:

$$\gamma = \frac{\Delta t_{\mathrm{obs}}}{\Delta\tau_{\mathrm{prop}}} = \frac{10\,\mathrm{years}}{4\,\mathrm{months}} = 30, \tag{7}$$

corresponding to $v = c\sqrt{1-\gamma^{-2}} \approx c(1-5.6\times10^{-4})$. Since $v < c$, the trajectory is strictly timelike and causality is preserved at every point.

2.2 The paradox

Now consider the moment $t = 5$ years in Bob's rest frame. At this moment, Bob begins building the wall at M.

In Alice's proper time, $t = 5$ years for Bob corresponds to $\tau = 2$ months for Alice (half of the total journey of 4 months), since the journey is symmetric. At $\tau = 2$ months, Alice is at the midpoint M of her journey — that is, at spatial position $M = L/2$.

The paradox: at $t = 5$ years (Bob's frame),

- The bricks are **on Alice's spacecraft**, at spatial position M, in transit toward Y.
- Bob is **building the wall at** M, using bricks that will be delivered at Y in another 5 years.

The bricks appear to be simultaneously:

1. at position M on Alice's spacecraft (proper time $\tau = 2$ months),
2. at position M in the wall Bob is building (coordinate time $t = 5$ years).

These are two distinct spacetime events with the same spatial coordinate $x = M$ but different proper times along the worldline of the bricks. No violation of causality occurs — but the standard Lorentz transformation, which assigns a unique spatial position to each moment of coordinate time, cannot accommodate this structure.

2.3 Resolution via non-injectivity

The paradox is resolved by recognising that the worldline of the bricks $X^\mu(\tau)$ is *non-injective* with respect to the simultaneity foliation $\{\Sigma_t\}$ of Bob's rest frame. Specifically, there exist two proper times $\tau_1 \neq \tau_2$ such that:

$$X^0(\tau_1) = X^0(\tau_2) = t^* = 5\,\mathrm{years}, \tag{8}$$

with corresponding spatial positions:

$$X^1(\tau_1) = M \quad \text{(Alice's spacecraft at } \tau_1 = 2\,\mathrm{months}), \tag{9}$$

$$X^1(\tau_2) = M \quad \text{(the bricks in the wall at } \tau_2 \text{ in the future loop)}. \tag{10}$$

The two intersections of the worldline with Σ_{t^*} correspond to $N = 2$ appearances of the same physical entity at the same spatial location but at different proper times. This is not a contradiction: the Ontological Identity Principle (Section 8) asserts that both appearances are manifestations of a single physical system whose identity is carried by the worldline $X^\mu(\tau)$, not by its instantaneous spatial position.

2.4 The control function

The paradox occurs with a precise geometric relationship between O, M, Y, and the Lorentz factor γ. Small deviations from this relationship — variations in γ, in the position of M, or in the construction time — dissolve the paradox because they break the exact winding condition $w(\Sigma_t) = 2$.

This sensitivity is encoded in the configuration function:

$$f(X_M, X_Y) = \Theta(\gamma\,\tau(M) - t(M))\,, \tag{11}$$

where Θ is the Heaviside step function, $\tau(M)$ is the proper time of the spacecraft when it reaches M, and $t(M)$ is the corresponding coordinate time of Bob. When $f = 1$, the system is in the Topological Overlap regime and $N \geq 2$. When $f = 0$, the worldline is injective and the standard Lorentz transformation applies. The function f is the binary indicator of worldline non-injectivity: it is the kinematic precursor of the existence function $f : \mathcal{M} \to \{0, 1\}$ of the DGQ formalism [2].

3 Explicit Coordinate Formulation of the Ziegelstein Gedankenexperiment

3.1 Setup with explicit coordinates

We place the origin of Bob's rest frame at $O = (t, x) = (0, 0)$. The destination is at $Y = (t, x) = (t_Y, L)$ with $L = c \cdot 10\,\text{years}$. The midpoint is $M = L/2$.

The spacecraft departs at $t = 0$ with velocity $v = c\sqrt{1 - 1/\gamma^2}$ and $\gamma = 30$ as in eq. (7).

The worldline of the bricks in Bob's frame is:

$$X^1(\tau) = v \cdot \gamma\tau, \qquad X^0(\tau) = \gamma\tau, \tag{12}$$

for uniform relativistic motion.

3.2 The two intersections with Σ_{t^*}

The wall is built at coordinate time $t^* = 5\,\text{years}$ in Bob's frame, at position $x = M = L/2$.

First intersection. The spacecraft worldline passes through $x = M$ at coordinate time:

$$t_1 = \frac{M}{v} = \frac{L/2}{v} \approx \frac{L/2}{c} = 5\,\text{years}, \tag{13}$$

since $v \approx c$ for $\gamma = 30$. The corresponding proper time is:

$$\tau_1 = \frac{t_1}{\gamma} = \frac{5\,\text{years}}{30} = 2\,\text{months}. \tag{14}$$

The first spacetime event is therefore:

$$\mathcal{E}_1 = (t_1,\, x_1) = (5\,\text{years},\, M). \tag{15}$$

Second intersection. By the Ziegelstein Gedankenexperiment construction, the bricks are also present at $x = M$ at $t^* = 5\,\text{years}$ in the wall. The wall is built using bricks that will be delivered at Y and then transported back to M: the return journey from Y to M takes 5 years in Bob's frame, so the bricks arrive at M (in the wall) exactly at $t^* = 5\,\text{years}$.

The corresponding proper time for this second appearance is τ_2, satisfying:

$$X^0(\tau_2) = t^* = 5\,\text{years}, \qquad X^1(\tau_2) = M, \tag{16}$$

with $\tau_2 \neq \tau_1$. The second spacetime event is:

$$\mathcal{E}_2 = (t_2,\, x_2) = (5\,\text{years},\, M). \tag{17}$$

Non-injectivity confirmed. Both events share the same coordinate time t^* and the same spatial position $x = M$, but correspond to different proper times $\tau_1 \neq \tau_2$:

$$X^0(\tau_1) = X^0(\tau_2) = t^*, \qquad X^1(\tau_1) = X^1(\tau_2) = M, \qquad \tau_1 \neq \tau_2. \tag{18}$$

This is the explicit coordinate demonstration of condition (8). The worldline intersects Σ_{t^*} at $N = 2$ distinct proper times.

3.3 The proper-time gap

The proper-time separation between the two appearances is:

$$\Delta\tau = \tau_2 - \tau_1 = \frac{t_Y - t^*}{\gamma} = \frac{10\,\text{years} - 5\,\text{years}}{30} = 2\,\text{months}, \tag{19}$$

since the return journey from Y to M takes 5 years in Bob's frame and is contracted by γ in Alice's frame. This proper-time gap is the direct input to the topological phase offset $\Phi_2 = \gamma^2 v \Delta\tau$ of eq. (45).

3.4 Summary of the two events

Appearance	**Bob's time t**	**Position x**	**Proper time τ**
First (spacecraft at M)	5 years	$M = L/2$	$\tau_1 = 2\,\text{months}$
Second (bricks in wall)	5 years	$M = L/2$	$\tau_2 = 4\,\text{months}$

Table 2. Explicit coordinate values of the two spacetime intersections in the Ziegelstein Gedankenexperiment. Both appearances share the same (t, x) but differ in proper time τ.

4 Derivation of the Topological Phase Offset

4.1 Geometric origin of Φ_n

We now derive the topological phase offset Φ_n from first principles, showing explicitly how the proper-time gap between intersections produces a spatial shift in the boosted frame.

Consider two intersections of the worldline with Σ_{t^*} at proper times τ_1 and τ_n. In the original frame (Bob's rest frame), both intersections share the same coordinate time t^* but may differ in x:

$$\mathcal{E}_1 = (t^*,\, x_1), \qquad \mathcal{E}_n = (t^*,\, x_n). \tag{20}$$

Applying the standard Lorentz boost to each event independently:

$$x_1' = \gamma(x_1 - vt^*), \tag{21}$$
$$x_n' = \gamma(x_n - vt^*). \tag{22}$$

The spatial separation in the boosted frame is:

$$x_n' - x_1' = \gamma(x_n - x_1). \tag{23}$$

4.2 Expressing $x_n - x_1$ in terms of proper time

Along the worldline, the spatial coordinate evolves as:

$$x(\tau) = x_1 + v\,\gamma\,(\tau - \tau_1), \tag{24}$$

for locally uniform motion near each intersection. At the n-th intersection $\tau = \tau_n$:

$$x_n - x_1 = v\,\gamma\,(\tau_n - \tau_1). \tag{25}$$

4.3 The phase offset formula

Substituting (25) into (23):

$$x_n' - x_1' = \gamma \cdot v\,\gamma\,(\tau_n - \tau_1) = \gamma^2 v\,(\tau_n - \tau_1). \tag{26}$$

We define the topological phase offset as the additional shift in the boosted frame relative to the naive single-sheet result:

$$\boxed{\Phi_n := x_n' - x_1' = \gamma^2 v\,(\tau_n - \tau_1).} \tag{27}$$

Remark 4.1. *Equation* (27) *confirms the formal definition* (44) *from first principles: both give* $\Phi_n = \gamma^2\,v\,(\tau_n - \tau_1)$*. The factor* γ^2 *arises from a double Lorentz amplification: once from the time dilation* $\Delta t = \gamma\Delta\tau$*, and once from the spatial boost* $\Delta x' = \gamma\Delta x$*. For* $v \approx c$ *and* $\gamma \gg 1$*, the factor* $\gamma^2 v \approx \gamma^2 c$ *correctly captures the relativistic amplification of the proper-time gap into a macroscopic spatial separation in the boosted frame —* 150 *light-years for the Ziegelstein Gedankenexperiment.*

4.4 Numerical value for the Ziegelstein Gedankenexperiment

Using $\gamma = 30$, $v \approx c$, and $\Delta\tau = \tau_2 - \tau_1 = 2\,\text{months}$ from eq. (19):

$$\Phi_2 = \gamma^2 v\,\Delta\tau \approx 900 \cdot c \cdot 2\,\text{months} = 1{,}800\,\text{light-months} = 150\,\text{light-years}. \tag{28}$$

This is the spatial separation between the two appearances of the bricks as measured in Alice's boosted frame. In Bob's frame both appearances are at $x = M$; in Alice's frame they are separated by 150 light-years — a macroscopic topological effect of the phase offset.

5 From Non-Injectivity to Winding: The Bijection Failure

5.1 Where the standard Lorentz transformation breaks down

The standard Lorentz boost (2) assigns to each event (t, x) a unique image $x' = \gamma(x - vt)$. This assignment is a bijection if and only if the map $\tau \mapsto X^0(\tau)$ is monotone, which is guaranteed for uniform motion.

For a non-uniformly moving worldline, the derivative:

$$\frac{dX^0}{d\tau} = \gamma(\tau)\, c > 0 \tag{29}$$

is always positive for a timelike worldline (since $\gamma(\tau) \geq 1$). This means $X^0(\tau)$ is always monotone in τ for any timelike worldline in Minkowski space.

The subtlety. Non-injectivity does not arise from $dX^0/d\tau = 0$ in Minkowski space. It arises from the *geometric folding* of the worldline relative to the simultaneity foliation $\{\Sigma_t\}$ when the trajectory involves a *turnaround* in the spatial direction — as in the twin paradox, but at the kinematic level of the foliation rather than the trajectory itself.

5.2 The foliation intersection condition

For a worldline that departs from O, reaches Y, and the bricks subsequently travel back toward M, the spatial component $X^1(\tau)$ is non-monotone: it increases from O to Y and then decreases from Y toward M.

The intersection of this worldline with the fixed hypersurface $\Sigma_{t^*} = \{(t^*, x) : x \in \mathbb{R}\}$ occurs wherever:

$$X^0(\tau) = t^* \tag{30}$$

Since $X^0(\tau)$ is monotone, this equation has in general *one* solution. But the spatial component $X^1(\tau)$ can take the value $x = M$ at two distinct proper times τ_1 and τ_2 if the worldline passes through M both on the outward journey and on the return journey.

The correct statement of non-injectivity is therefore:

$$\exists\, \tau_1 \neq \tau_2 : \quad X^0(\tau_1) = X^0(\tau_2) = t^* \text{ and } X^1(\tau_1) = X^1(\tau_2) = M. \tag{31}$$

This is a condition on the *full four-vector* $X^\mu(\tau)$, not just on $X^0(\tau)$ alone.

5.3 The winding number and the function f

The winding number of the worldline through Σ_{t^*} at position M is defined as the number of solutions to (31):

$$w(t^*, M) := \#\left\{\tau : X^0(\tau) = t^*,\ X^1(\tau) = M\right\}. \tag{32}$$

For the Ziegelstein Gedankenexperiment, $w(t^*, M) = 2$.

The configuration function f of eq. (11) is the indicator of the regime $w \geq 2$:

$$f(X_M, X_Y) = 1 \iff w(t^*, M) \geq 2 \iff \gamma \geq \gamma_{\text{crit}}. \tag{33}$$

The connection between $f = 1$, $w = 2$, and $\gamma \geq \gamma_{\text{crit}}$ is now established explicitly: all three conditions are equivalent statements of worldline non-injectivity.

5.4 Precision of the paradox

The Ziegelstein Gedankenexperiment requires a precise geometric relationship between the distances L, the construction time t^*, and the Lorentz factor γ. Specifically, the condition $w(t^*, M) = 2$ requires:

$$\frac{t^*}{t_Y/2} = 1, \qquad t_Y = \frac{L}{v} \approx \frac{L}{c}, \tag{34}$$

i.e. the construction time equals exactly half the total journey time. A deviation δt^* from this precise ratio gives:

$$w(t^* + \delta t^*, M) = \begin{cases} 2 & \text{if } |\delta t^*| < \delta_{\max}, \\ 1 & \text{if } |\delta t^*| \geq \delta_{\max}, \end{cases} \tag{35}$$

where $\delta_{\max} \sim L/(c\gamma^2)$ is the precision window. For $\gamma = 30$ and $L = 10$ light-years, $\delta_{\max} \sim 4$ days. The paradox is therefore not a mathematical coincidence but a robust phenomenon within a finite precision window, falsifiable by varying t^* and observing the transition from $w = 2$ to $w = 1$.

6 Standard Lorentz Transformations and Their Hidden Assumption

6.1 The injectivity assumption

The standard Lorentz boost (1)–(3) is derived under the assumption that the transformation is a bijection on Minkowski space. This is guaranteed if and only if the worldline satisfies condition (4).

To see this, note that the inverse transformation

$$t = \gamma\Big(t' + \frac{v}{c^2}x'\Big), \tag{36}$$

$$x = \gamma(x' + vt'), \tag{37}$$

assigns a unique pre-image (t, x) to each (t', x') if and only if the map $\tau \mapsto (t(\tau), x(\tau))$ is injective. If injectivity fails, multiple events $(t, x_1), (t, x_2), \ldots$ in the original frame map to events with the same coordinate time t' in the boosted frame, but the inverse transformation recovers only one of them.

6.2 When injectivity fails

Consider a worldline $X^\mu(\tau)$ parametrised by proper time τ. The coordinate time component is:

$$X^0(\tau) = \gamma\tau + \text{const}, \tag{38}$$

for uniform motion. This is monotone in τ, so injectivity holds.

For non-uniform motion — specifically, for trajectories that involve turning points, oscillations, or geometric folding relative to Σ_t — the function $\tau \mapsto X^0(\tau)$ can become non-monotone. The critical condition is:

$$\frac{dX^0}{d\tau} = 0 \quad \text{at some } \tau = \tau_*, \tag{39}$$

which in the radial AdS picture corresponds to the worldline reaching the turning point of its oscillation at $z = z_{\max}$.

At sufficiently large γ, the geometric compression of the worldline relative to Σ_t produces exactly this condition: the worldline "folds" over Σ_t and intersects it multiple times.

6.3 The winding number

The multiplicity of intersections is captured by the winding number of the worldline through Σ_t:

$$w(\Sigma_t) = \oint_{\partial\Sigma} dX^\mu \tag{40}$$

For an injective worldline, $w = 1$. For a non-injective worldline with N intersections, $w = N$. The Bricks Paradox corresponds to $w = 2$.

7 Extended Lorentz Transformations

7.1 Definition

Definition 7.1 (Extended Lorentz Transformation). *Let $X^\mu(\tau)$ be a worldline satisfying condition* (8) *with $N \geq 2$ intersections $\{(t^*, x_n)\}_{n=1}^N$ of Σ_{t^*}. The* extended Lorentz transformation *associated to this worldline is the set of N maps:*

$$t'_n = \gamma\Big(t - \frac{v}{c^2}x_n\Big), \tag{41}$$

$$x'_n = \gamma(x_n - vt) + \Phi_n, \tag{42}$$

$$y'_n = y, \quad z'_n = z, \tag{43}$$

for $n = 1, \ldots, N$, where Φ_n is the topological phase offset *of the n-th sheet, defined by:*

$$\Phi_n := \gamma^2\, v\, (\tau_n - \tau_1)\,, \tag{44}$$

with τ_n the proper time of the n-th intersection.

Remark 7.2. *For $N = 1$, equation* (42) *reduces to the standard Lorentz boost* (2) *with $\Phi_1 = 0$. The extended transformation is therefore a genuine generalisation: it contains the standard result as a special case.*

7.2 Physical meaning of the phase offset

The topological phase offset Φ_n measures the proper-time displacement between the n-th intersection and the reference intersection $n = 1$, weighted by the kinematic factor γv.

Geometrically, Φ_n is the spatial separation between the n sheets of the worldline as measured in the boosted frame. It is not a free parameter: it is entirely determined by the worldline $X^\mu(\tau)$ and the Lorentz factor γ.

For the Bricks Paradox with $N = 2$:

$$\Phi_2 = \gamma^2\, v\, (\tau_2 - \tau_1), \tag{45}$$

where $\tau_2 - \tau_1$ is the proper-time separation between the two appearances of the bricks at position M. This is the precise mathematical statement of the paradox: the two appearances are separated by a topological phase Φ_2 in the boosted frame, not by a spatial distance in the rest frame.

7.3 The sheet-symmetric limit

By the Ontological Identity Principle (Section 8), all N sheets are manifestations of the same physical entity. This imposes a constraint on the extended transformations: physical observables must be independent of the sheet index n.

Define the sheet-averaged coordinate:

$$\bar{x}' := \frac{1}{N}\sum_{n=1}^{N} x'_n = \gamma(\bar{x} - vt) + \frac{1}{N}\sum_{n=1}^{N} \Phi_n, \tag{46}$$

where $\bar{x} = N^{-1}\sum_n x_n$ is the centroid of the N intersections. The sheet-averaged transformation coincides with the standard Lorentz boost applied to the centroid, plus a mean topological phase $\bar{\Phi} = N^{-1}\sum_n \Phi_n$.

In the DGQ framework, the sheet-averaged Pauli operators $\hat{\Sigma}_\alpha = N^{-1}\sum_i \sigma_\alpha^{(i)}$ are exactly the quantum-mechanical realisation of this sheet-averaging [2]. The extended Lorentz transformation is therefore the kinematic origin of the sheet-symmetric operator structure of the DGQ.

8 The Ontological Identity Principle

8.1 Statement

The N spatial intersections $\{x_n(t)\}_{n=1}^N$ of the worldline with Σ_t do not represent N distinct physical entities. They are N *appearances* of a single entity whose identity is carried by the continuous worldline $X^\mu(\tau) \in C^\infty(\mathbb{R})$, not by its instantaneous spatial position.

Formally, any local unitary operator U_n acting on the n-th intersection satisfies:

$$U_n \equiv U_m, \quad \forall\, n, m \in \{1, \ldots, N\}, \tag{47}$$

because topological continuity of $X^\mu(\tau)$ implies that any operation applied at one intersection propagates coherently to all others.

8.2 Energy-momentum conservation

The N intersections produce an effective energy-momentum density:

$$T^{00}_{\text{eff}}(x, t) = \sum_{n=1}^{N} m\, \delta^3(x - x_n(t)), \tag{48}$$

where m is the rest mass of the single physical entity. The integrated energy $\int T^{00}_{\text{eff}} d^3x = Nm$ appears to violate conservation.

The resolution is exact. Since all N intersections are segments of the same continuous worldline, the divergence condition

$$\partial_\mu T^{\mu\nu}_{\text{eff}} = 0 \tag{49}$$

holds identically, as verified by the chain rule along $X^\mu(\tau)$:

$$\partial_0 T^{00}_{\text{eff}} + \partial_j T^{j0}_{\text{eff}} = \sum_{n=1}^{N} m\left[\dot{x}^j_n(t)\,\frac{\partial}{\partial x^j}\delta^3(x - x_n(t)) + \frac{\partial}{\partial t}\delta^3(x - x_n(t))\right] = 0. \tag{50}$$

The apparent energy surplus $(N-1)m$ is a consequence of the foliation sampling the worldline N times, not a creation of mass.

9 Conservation Laws under Extended Lorentz Transformations

Theorem 9.1 (Conservation under ELT)**.** *Under the extended Lorentz transformation* (41)–(43)*, all standard conservation laws — energy-momentum, charge, and angular momentum — are preserved, provided the sheet average is taken over all N intersections.*

Proof. **Energy-momentum.** The total four-momentum of the system is:

$$P^\mu_{\text{total}} = \sum_{n=1}^{N} m\, u^\mu(\tau_n), \tag{51}$$

where $u^\mu(\tau_n) = dX^\mu/d\tau|_{\tau_n}$ is the four-velocity at the n-th intersection. By the Ontological Identity Principle, $u^\mu(\tau_n)$ is the four-velocity of the *same* physical entity at different proper times; the sum counts it N times. The physically observable four-momentum is the sheet average:

$$\bar{P}^\mu = \frac{1}{N}\sum_{n=1}^{N} m\, u^\mu(\tau_n) = m\, \bar{u}^\mu, \tag{52}$$

which transforms under the standard Lorentz law $\bar{P}^\mu \to \Lambda^\mu{}_\nu \bar{P}^\nu$, as required.

Charge. By equation (50), $\partial_\mu J^\mu = 0$ holds for the effective current $J^\mu = q\, T^{0\mu}_{\text{eff}}/m$. The extended transformation preserves the current conservation law because it preserves the worldline $X^\mu(\tau)$ and the Ontological Identity.

Angular momentum. The angular momentum tensor $M^{\mu\nu} = x^\mu P^\nu - x^\nu P^\mu$ is computed using the sheet-averaged position $\bar{x}^\mu$ and momentum $\bar{P}^\mu$, which transform covariantly under (41)–(43) after averaging. □

10 The Critical Lorentz Factor

10.1 Geometric derivation

The transition from injective to non-injective worldlines occurs at a critical Lorentz factor γ_{crit} determined by the geometry of the trajectory and the simultaneity foliation.

For a worldline oscillating radially in AdS with amplitude $z_{\max} \sim L_{\text{AdS}}$ and near-boundary cutoff ϵ, the radial oscillation wavelength in boundary coordinates is:

$$\lambda_{\text{rad}} = \frac{L_{\text{AdS}}}{\gamma}. \tag{53}$$

Non-injectivity activates when $\lambda_{\text{rad}} < \epsilon$, i.e. when the oscillation wavelength is smaller than the minimum resolvable boundary cell. This gives:

$$\gamma_{\text{crit}} = \frac{L_{\text{AdS}}}{\epsilon} = \frac{L_{\text{AdS}}}{\tau_* R_B}, \tag{54}$$

where $\tau_* = \sqrt{1 + a^2/R_B^2} - a/R_B$ is the geometric tangency parameter of the companion paper [1].

10.2 Numerical value

For the concrete parameters of [2]:

$$\gamma = \frac{\Delta t_{\mathrm{obs}}}{\Delta \tau_{\mathrm{prop}}} = \frac{1.57 \times 10^{8}\,\mathrm{s}}{7.2 \times 10^{3}\,\mathrm{s}} \approx 21{,}915, \tag{55}$$

corresponding to $v \approx c(1 - 10^{-9})$. This is the minimum Lorentz factor for which the worldline non-injectivity activates in the dynamic implementation of the DGQ.

10.3 The Bricks Paradox threshold

For the Bricks Paradox of Section 2, the critical condition is:

$$\gamma_{\mathrm{bricks}} = \frac{\Delta t_{\mathrm{obs}}}{\Delta \tau_{\mathrm{prop}}} = \frac{10\,\mathrm{years}}{4\,\mathrm{months}} = 30. \tag{56}$$

This is the minimum Lorentz factor for which the bricks worldline intersects Σ_{t^*} in $N = 2$ distinct spatial points, producing the paradox. For $\gamma < 30$, the worldline is injective and no paradox occurs. For $\gamma = 30$ exactly, the paradox is maximal: the two appearances of the bricks coincide at the same spatial position $x = M$. For $\gamma > 30$, the two appearances are spatially separated and $N > 2$.

11 Connection to the DGQ Framework

11.1 Multi-sheet Hilbert space from ELT

The extended Lorentz transformation (41)–(43) produces N distinct coordinate patches $\{(t'_n, x'_n)\}_{n=1}^{N}$ for each event in the original frame. Quantising this structure — replacing each classical sheet with a Hilbert space $\mathcal{H}_n$ — yields the multi-sheet Hilbert space:

$$\mathcal{H}_{\mathrm{DG}} = \bigotimes_{n=1}^{N} \mathcal{H}_n, \tag{57}$$

which is the fundamental arena of the De Giuseppe Qubit [2].

The sheet-symmetric operators $\hat{\Sigma}_\alpha = N^{-1} \sum_n \sigma_\alpha^{(n)}$ are the quantum realisation of the sheet-averaged coordinates (46), and the Ontological Identity Principle (47) is the quantum analogue of the classical conservation law (49).

11.2 Non-injectivity as necessary and sufficient condition

The companion paper [1] proves that worldline non-injectivity — the condition that the extended Lorentz transformation has $N > 1$ branches — is both necessary and sufficient for the existence of a finite holographic spacetime with finite Ryu–Takayanagi entanglement entropy.

The extended Lorentz transformation is therefore not merely a kinematic curiosity. It is the microscopic mechanism from which spacetime geometry, holographic entanglement, and quantum computational structure all emerge.

11.3 The N-sheet intersection number

The number of branches N of the extended Lorentz transformation is related to the holographic UV cutoff ϵ by:

$$N(\epsilon) \sim \frac{1}{\epsilon^{d-2}}, \tag{58}$$

which is precisely the scaling of Lemma 2 of [1]. The extended Lorentz transformation thus encodes, at the kinematic level, the topological averaging mechanism that regularises holographic UV divergences.

12 The Ziegelstein Mechanism as Universal UV Regulator

12.1 The UV divergence problem

The Ziegelstein Gedankenexperiment is not merely a kinematic curiosity. The same geometric mechanism that produces the two appearances of the bricks at position M is precisely the mechanism that regulates the ultraviolet divergences of holographic quantum gravity.

In holographic theories, the Ryu–Takayanagi entanglement entropy of a boundary region A diverges as the UV cutoff $\epsilon \to 0$ [6]:

$$S_{\mathrm{RT}} = \frac{\mathrm{Area}(\gamma_B)}{4G_N} \sim \frac{1}{\epsilon^{d-2}}, \qquad \epsilon \to 0. \tag{59}$$

This divergence is ordinarily regulated by an external, *ad hoc* cutoff whose physical origin is left unexplained. The extended Lorentz transformation provides that physical origin.

12.2 How non-injectivity cancels the divergence

When the worldline is non-injective with $N(\epsilon)$ intersections, the physically observable entanglement entropy is not the single-sheet value but the topological average over all N sheets [1]:

$$S_{\mathrm{DG}} := \frac{1}{N(\epsilon)} \sum_{i=1}^{N(\epsilon)} \frac{\mathrm{Area}(\gamma_{A,i})}{4G_N}. \tag{60}$$

The intersection multiplicity scales as (Lemma 2 of [1]):

$$N(\epsilon) \sim \frac{1}{\epsilon^{d-2}}, \tag{61}$$

for the same geometric reason that produces the $N = 2$ appearances in the Ziegelstein Gedankenexperiment: the worldline is compressed relative to the simultaneity foliation by exactly the factor $\epsilon^{-(d-2)}$ needed to match the UV divergence degree for degree.

By the Ontological Identity Principle, all N minimal surfaces $\gamma_{A,i}$ share the same near-boundary geometry, so each area scales as $\epsilon^{-(d-2)}$. Substituting into (60):

$$S_{\mathrm{DG}} \sim \frac{1}{N(\epsilon)} \cdot N(\epsilon) \cdot \frac{1}{\epsilon^{d-2}} = \frac{1}{\epsilon^{d-2}} \cdot \epsilon^{d-2} = \mathcal{O}(1). \tag{62}$$

The UV divergence is exactly cancelled. No external cutoff is introduced. The regulator is the topology of the worldline itself.

12.3 The algebraic identity behind the cancellation

The cancellation (62) is the holographic instance of a single algebraic identity:

$$\boxed{N(\epsilon) \cdot \epsilon^{d-2} = \mathcal{O}(1).} \tag{63}$$

This identity holds because the intersection multiplicity $N(\epsilon)$ and the UV divergence $\epsilon^{-(d-2)}$ are both controlled by the same geometric object: the $(d-2)$-dimensional transverse volume of the boundary cell, which sets simultaneously the holographic entropy density and the worldline winding rate.

In the language of the extended Lorentz transformation, this identity is the statement that the topological phase offsets $\{\Phi_n\}_{n=1}^N$ distribute the UV weight of the entanglement entropy uniformly across N sheets, producing a finite sheet-averaged result from individually divergent contributions.

12.4 Universality: three levels of the same cancellation

The companion paper [1] proves that the identity (63) operates identically at three distinct levels of physical theory:

Level 1: Holographic entanglement. The RT entropy $S_{\mathrm{RT}} \sim \epsilon^{-(d-2)}$ is regulated to $S_{\mathrm{DG}} = \mathcal{O}(1)$ by the topological average (60). The physically observable entanglement entropy is finite without any external cutoff.

Level 2: Classical electrodynamics. For a non-injective worldline with $\gamma > \gamma_{\mathrm{crit}}$, the standard Lienard–Wiechert retarded potential admits $N(\epsilon)$ distinct retarded events. The Coulomb self-energy of each sheet diverges as $\mathcal{E}_{(i)} \sim \epsilon^{-(d-2)}$, but the topological average:

$$\langle \mathcal{E} \rangle_{\mathrm{DG}} := \frac{1}{N(\epsilon)} \sum_{i=1}^{N(\epsilon)} \mathcal{E}_{(i)} \sim \epsilon^{-(d-2)} \cdot \epsilon^{d-2} = \mathcal{O}(1), \tag{64}$$

is finite. The Larmor power is unchanged. Maxwell's equations in their standard form are recovered — their own UV regulator is the topology of the worldline.

Level 3: Gravitational singularities. Near a classical spacetime singularity, the gravitational energy density diverges as $\rho_{\mathrm{sing}} \sim \epsilon^{-(d-2)}$. Under non-injectivity, the topological average:

$$\langle \rho \rangle_{\mathrm{DG}} := \frac{1}{N(\epsilon)} \sum_{i=1}^{N(\epsilon)} \rho^{(i)}(\epsilon) \sim \mathcal{O}(1), \tag{65}$$

is finite. The Big Bang is not a moment of infinite density but a topological phase transition from $N = 1$ to $N > 1$ — from no spacetime to finite spacetime. The Schwarzschild singularity is replaced by a region of finite curvature with topological structure $N(r) \sim r^{-(d-2)}$.

12.5 Comparison table

Level	Holographic	Electromagnetic	Gravitational
UV-divergent object	RT area $\sim \epsilon^{-(d-2)}$	Coulomb energy $\sim \epsilon^{-(d-2)}$	Energy density $\sim \epsilon^{-(d-2)}$
Multiplicity	$N(\epsilon) \sim \epsilon^{-(d-2)}$	$N(\epsilon) \sim \epsilon^{-(d-2)}$	$N(\epsilon) \sim \epsilon^{-(d-2)}$
Regulated quantity	$S_{\rm DG} = \mathcal{O}(1)$	$\langle \mathcal{E} \rangle_{\rm DG} = \mathcal{O}(1)$	$\langle \rho \rangle_{\rm DG} = \mathcal{O}(1)$
Observable unchanged	Modular Hamiltonian	Larmor power	Einstein equations
Free parameters	None	None	None
Cancellation	$N \cdot \epsilon^{d-2} = \mathcal{O}(1)$	$N \cdot \epsilon^{d-2} = \mathcal{O}(1)$	$N \cdot \epsilon^{d-2} = \mathcal{O}(1)$

Table 3. The Ziegelstein mechanism operates identically at three levels of physical theory. The algebraic identity $N(\epsilon) \cdot \epsilon^{d-2} = \mathcal{O}(1)$ is the universal regulator at each level.

12.6 From dinner table to UV regulation

The Ziegelstein Gedankenexperiment was conceived as a thought experiment about bricks appearing simultaneously in two places. The resolution — that both appearances are manifestations of the same worldline, distributed across two topological sheets by the extended Lorentz transformation — turns out to be precisely the mechanism that distributes UV divergences across $N(\epsilon)$ sheets and renders them finite.

The bricks at position M at $t^* = 5$ years are the $N = 2$ instance of the universal regulator (63). The wall being built is not a paradox. It is the first example of UV regulation by worldline non-injectivity, seen at the scale of everyday objects.

Remark 12.1. *The cancellation* (62) *requires no new physics beyond special relativity and the holographic dictionary. The only input is the extended Lorentz transformation derived in Section 7: the existence of $N > 1$ branches of the transformation, with topological phase offsets $\Phi_n = \gamma^2 v(\tau_n - \tau_1)$, is sufficient to render all three classes of UV divergence finite. This is the precise sense in which the Ziegelstein Gedankenexperiment is not merely a kinematic curiosity but the seed of a universal geometric principle.*

13 The Injective Limit and Recovery of Standard Relativity

Theorem 13.1 (Recovery of Standard LT). *In the limit $N \to 1$ (injective worldline, $\gamma < \gamma_{\rm crit}$), the extended Lorentz transformation* (41)–(43) *reduces exactly to the standard Lorentz boost* (1)–(3).

Proof. For $N = 1$, there is a single intersection $\tau_1 = \tau$ and a single phase offset $\Phi_1 = 0$

by definition (44). Equations (41)–(43) reduce to:

$$t'_1 = \gamma\Big(t - \frac{v}{c^2}x\Big), \tag{66}$$

$$x'_1 = \gamma(x - vt) + 0 = \gamma(x - vt), \tag{67}$$

$$y'_1 = y, \quad z'_1 = z, \tag{68}$$

which are exactly (1)–(3). □

Remark 13.2. *The standard Lorentz transformation is therefore a special case of the extended transformation, valid in the regime $\gamma < \gamma_{\mathrm{crit}}$ where the injectivity assumption* (4) *holds. Special relativity is not modified or contradicted: it is extended to a larger kinematic regime.*

14 Physical Consequences

14.1 Spontaneous synchronisation

The N sheets of the extended Lorentz transformation are not independent: they are connected by the Ontological Identity Principle. Any operation applied to one sheet propagates coherently to all others via (47). This produces a *spontaneous synchronisation* of the N appearances — a collective behaviour that has no analogue in standard special relativity.

In the Bricks Paradox, this means that the physical state of the bricks — their mass, charge, quantum state — is the same at both appearances. The wall being built at M and the spacecraft in transit are not two different objects: they are one object in two kinematic appearances.

14.2 Emergent entanglement

In the quantum regime, the N sheets of (57) are intrinsically entangled by the Ontological Identity: the state of the system on sheet n is not independent of the state on sheet m. This intrinsic entanglement requires no external entangling gate — it is a direct consequence of the kinematic structure of the extended Lorentz transformation. This is the origin of the emergent entanglement of the DGQ [2].

14.3 Topological protection

The phase offsets Φ_n in (42) are topological invariants: they cannot be continuously deformed to zero without crossing the critical threshold $\gamma = \gamma_{\mathrm{crit}}$. This topological rigidity is the kinematic origin of the decoherence suppression $\Gamma_{\mathrm{DG}} \sim \Gamma_{\mathrm{single}}/N$ derived in [2].

15 Summary Table

Property	**Standard LT** ($N = 1$)	**Extended LT** ($N > 1$)
Injectivity	$\tau \mapsto X^0(\tau)$ injective	$\tau \mapsto X^0(\tau)$ non-injective
Image of a single event	One point (t', x')	N points (t'_n, x'_n)
Phase offsets	$\Phi_1 = 0$	$\Phi_n = \gamma^2 v(\tau_n - \tau_1)$
Ontological structure	Single appearance	N appearances of one entity
Energy-momentum	$P^\mu = mu^\mu$	$\bar{P}^\mu = m\bar{u}^\mu$ (averaged)
Conservation laws	Standard	Preserved via sheet average
Winding number	$w = 1$	$w = N$
Hilbert space	$\mathcal{H}$	$\bigotimes_{n=1}^{N} \mathcal{H}_n$
Entanglement	Requires external gates	Intrinsic (Ontological Identity)
Regime	$\gamma < \gamma_{\text{crit}}$	$\gamma > \gamma_{\text{crit}}$

Table 1. Comparison between standard and extended Lorentz transformations.

16 Conclusions

We have derived the extended Lorentz transformations that govern the kinematic regime of worldline non-injectivity, where the standard boost $x' = \gamma(x - vt)$ must be replaced by N branches $x'_n = \gamma(x_n - vt) + \Phi_n$.

The physical origin of the extended structure was illustrated by the Bricks Paradox: a thought experiment in which a single set of physical objects appears simultaneously at two spacetime locations without any violation of causality or energy conservation. The paradox is not a contradiction but a physical manifestation of worldline non-injectivity, resolved by the Ontological Identity Principle.

The extended Lorentz transformations are the kinematic foundation of three interconnected structures:

1. The multi-sheet Hilbert space of the De Giuseppe Qubit [2], whose sheet-symmetric operators arise from the sheet averaging of the extended transformation.

2. The non-injectivity theorem of [1], which proves that $N > 1$ is necessary and sufficient for finite holographic spacetime.

3. The topological regularisation mechanism that cancels UV divergences in the Ryu–Takayanagi entropy, the Coulomb self-energy, and gravitational singularities by the same algebraic identity $N(\epsilon) \cdot \epsilon^{d-2} = O(1)$.

All three structures were seeded by the same physical intuition: the Bricks Paradox, a dinner conversation in which bricks appeared simultaneously in transit and in a wall, leading to the recognition that worldline non-injectivity is not a kinematic accident but the geometric engine of spacetime formation.

Declarations

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